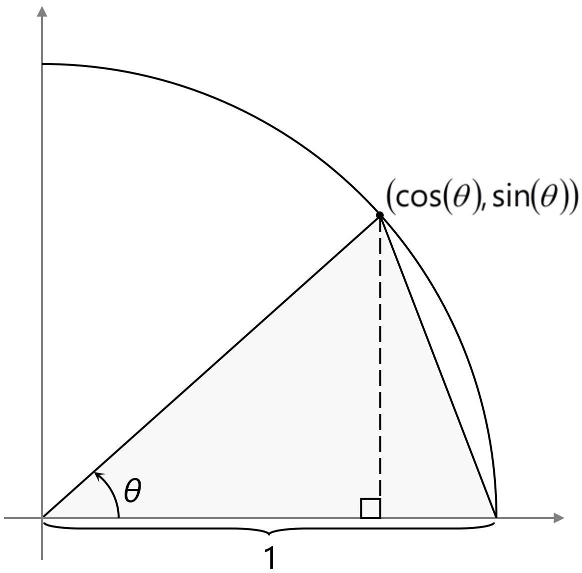
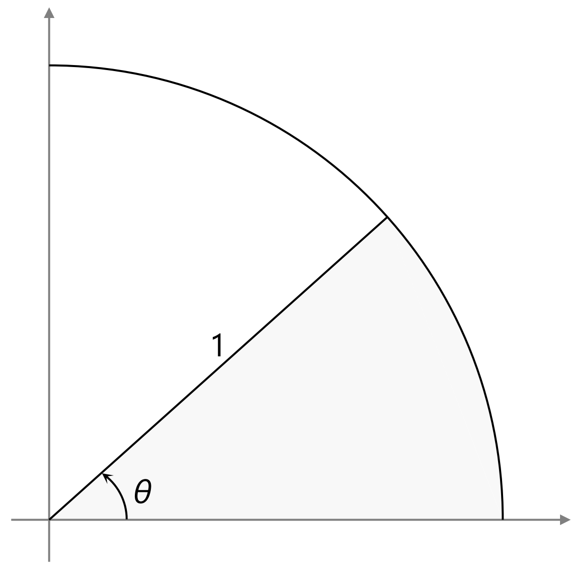
From the Unit Circle…



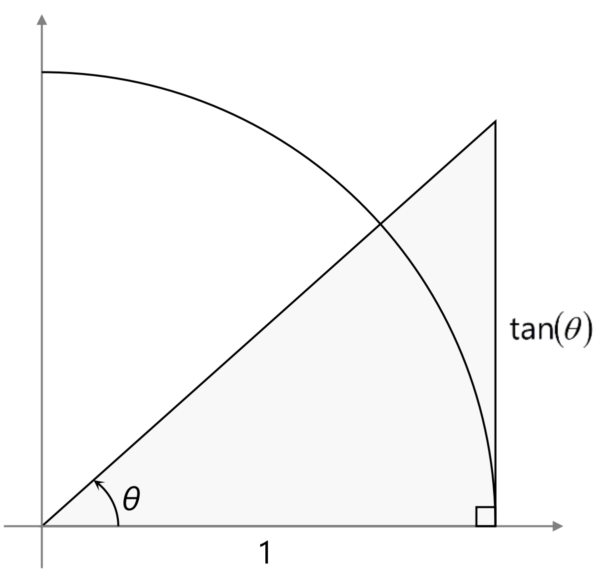
1. The area of this right triangle is

   $\frac{1}{2}bh = \frac{1}{2}\cdot 1\cdot \sin(\theta) = \frac{1}{2}\cdot \sin(\theta)$



2. The area of this circular sector is

   $\frac{1}{2}r^2\theta = \frac{1}{2}\cdot 1^2\cdot \theta = \frac{1}{2}\theta$



3. The area of this right triangle is

   $\frac{1}{2}bh = \frac{1}{2}\cdot 1\cdot \tan(\theta) = \frac{1}{2}\cdot \tan(\theta)$

It should be clear that the areas described can be “ranked” in *ascending* order as presented;

that is: $\frac{1}{2}\cdot \sin(\theta) \;\leq\; \frac{1}{2}\theta \;\leq\; \frac{1}{2}\cdot \tan(\theta)$

We may rewrite this as: $\frac{1}{2}\cdot \sin(\theta) \;\leq\; \frac{1}{2}\theta \;\leq\; \frac{1}{2}\cdot \frac{\sin(\theta)}{\cos(\theta)}$

Clearly, the common factor ½ can be disregarded, so we have: $\sin(\theta) \;\le\; \theta \;\le\; \dfrac{\sin(\theta)}{\cos(\theta)}$

Now, divide all terms by $\sin(\theta)$:

$$\frac{\sin(\theta)}{\sin(\theta)} \;\le\; \frac{\theta}{\sin(\theta)} \;\le\; \frac{\sin(\theta)}{\cos(\theta)} \Big/ \sin(\theta) \qquad \Rightarrow \qquad 1 \;\le\; \frac{\theta}{\sin(\theta)} \;\le\; \frac{1}{\cos(\theta)}$$

We assume that the sine is positive, so the inequality "directions" are preserved.

Next, though, we ***reciprocate*** all terms, and this <u>will</u> "reverse" the directions of each inequality:

$$1 \;\le\; \frac{\theta}{\sin(\theta)} \;\le\; \frac{1}{\cos(\theta)} \qquad \Rightarrow \qquad 1 \;\ge\; \frac{\sin(\theta)}{\theta} \;\ge\; \cos(\theta)$$

Therefore the ratio $\dfrac{\sin(\theta)}{\theta}$ is "trapped" between 1 and the value of $\cos(\theta)$

We may write the relationships in reverse, as $\cos(\theta) \;\le\; \dfrac{\sin(\theta)}{\theta} \;\le\; 1$

Interestingly, the relationships hold for $|\theta| \le \pi/2$ due to the even & odd properties of cosine and sine, respectively. With this in mind, we finally let θ approach 0 in each expression.

If these limits exist, we should have $\lim\limits_{\theta\to 0}(\cos(\theta)) \;\le\; \lim\limits_{\theta\to 0}\left(\dfrac{\sin(\theta)}{\theta}\right) \;\le\; \lim\limits_{\theta\to 0}(1)$

We can quickly agree that $\lim\limits_{\theta\to 0}(\cos(\theta)) = 1$ and $\lim\limits_{\theta\to 0}(1) = 1$

Therefore, since $\dfrac{\sin(\theta)}{\theta}$ is between these two limited values, we conclude that $\lim\limits_{\theta\to 0}\left(\dfrac{\sin(\theta)}{\theta}\right) = 1$ as well.

$$1 \;\le\; \lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) \;\le\; 1$$

The limits of the quantities which "trapped" $\dfrac{\sin(\theta)}{\theta}$ are equal...so the limit of $\dfrac{\sin(\theta)}{\theta}$ is "squeezed" between those to ***<u>share the same limit</u>.***

...it's the <u>Squeeze Theorem</u>!